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BECHMANN'S NUMBER FOR HARMONIC OVERTONES OF THICKNESS-SHEAR VIBRATIONS OF ROTATED-Y-CUT QUARTZ PLATES

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Abstract—A method is developed for calculating Bechmann's Number for the harmonic overtones of antisymmetric thickness-shear vibrations of crystal plates. In most cases the solution can be expressed as a simple, explicit formula. Values of Bechmann's Number for the AT-cut-quartz plate for the first ten modes are calculated for the ranges of percentage plate-back encountered in applications.

INTRODUCTION

THE rotated-Y-cut [1] quartz crystal plate contains a digonal axis of symmetry of the crystal, and it is cut at an angle (Θ) with the trigonal axis. When referred to rectangular axis in and normal to the plane of the plate, the constitutive relations exhibit monoclinic symmetry. For straight-crested waves propagating in the direction of the digonal axis (x_1) in the plane of the plate, thickness-shear motion is coupled not only with thickness-stretch but with face-shear as well; i.e. even though the major displacement component (u_1) is parallel to the x_1 axis, there will always be some accompanying displacement in the directions of the x_2 and x_3 axes, as shown by Ekstein [2].

Antisymmetric thickness-shear waves can easily be excited piezoelectrically in such plates [3] by an electric field across the thickness. These waves are characterized by an odd number of nodes of displacement u_1 through the thickness of the plate. At long wave lengths in an infinite plate, the frequencies are approximately proportional to the number of nodes: hence the term harmonic overtones.

In all but some very rare harmonic modes, the frequency has a minimum value (cut-off) at zero wave number (infinite wave length), below which waves exist but are non-propagating and the wave number is imaginary [4]. If there is a small mass loading due to electrode deposits on the faces of the plate, the frequencies of this "plated" plate are lowered, slightly, in proportion to the amount of mass loading.

Considered in this paper is an infinite plate with uniform electrodes on both faces of its central portion in the form of strips extending indefinitely in the direction of the x_3 axis but of finite width ($2a$) in the x_1 direction (see Fig. 1). With such a partially plated plate excited in thickness-shear vibration, mechanical energy can be trapped under the electrodes in the form of standing waves reflected at the interfaces $x_1 = \pm a$ [5]. The amplitude of the thickness-shear wave decays exponentially away from each interface in the unplated portion of the plate. Such a phenomenon is possible when the frequency is in the narrow range between the two cut-off frequencies.

In this frequency range, anharmonic overtones, corresponding to phase reversals of displacement under the electrodes in the x_1 direction, can also be trapped. When a finite

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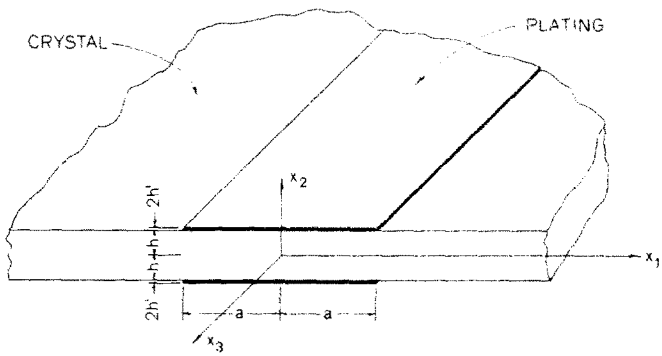

FIG. 1. Partially plated infinite plate.

crystal, for which this infinite plate is an idealization, is used in filter circuits, these overtones are a major source of unwanted responses. Bechmann observed that all but the fundamental mode could substantially be eliminated if the ratio of the electrode area to the total plate area is small enough [6].

Demands of the electrical circuit call for the largest possible electrode width; hence the need for a design criterion giving the largest width for which only the fundamental frequency of the anharmonic overtone series occurs between the two cut-off frequencies. In a plate such as described in the present paper, Bechmann's Number is defined as the ratio (a/h) of this critical plating width $(2a)$ to the plate thickness $(2h)$. Theoretical values have been found previously for the cases of the first thickness-shear harmonic mode [7] and for all the harmonic modes of thickness-twist waves propagating perpendicular to the digonal axis [8]. Empirical formulas for Bechmann's Number for harmonic overtones of thickness-twist and thickness-shear have been presented by Shockley, Curran, and Koneval [9]. In [7] it was shown that the criterion for Bechmann's Number for the first thickness-shear mode is that the plating width be equal to the wave length in the infinite, plated plate at the cut-off frequency of the corresponding mode in the unplated plate. This criterion was adopted as a postulate, in Ref. [8], for harmonic overtones of thickness-twist modes. In the present paper, the same criterion is proved to be valid, under certain restrictions, for all harmonic modes of thickness-shear.

To find these critical wave lengths, the frequency-wave number relationship, i.e. the dispersion relation, is required for the infinite, plated plate at long wave lengths. It is shown that the branches of the dispersion relation for an infinite, plated plate have very nearly the same shape as the corresponding branches for the unplated plate, and all frequencies are lowered by an amount nearly equal to the difference between the two cut-off frequencies. Except where another branch (e.g. thickness-stretch) is close by, these branches are sufficiently described by their ordinates, slopes, and curvatures at cut-off as predicted previously [8]. Therefore, a simple shift of the asymptotic description of the thickness-shear branches for the unplated plate, first developed by Ekstein [2], is all that is necessary, in most cases, for the plated plate.

The two asymptotic equations are then employed to obtain a simple, explicit formula for Bechmann's Number, and values are computed for the AT-cut $(\Theta = 35° 15')$ quartz plate for the first ten modes.

GENERAL EQUATIONS

With the usual strain–displacement relations

$$
\begin{aligned}
S_1 &= u_{1,1}, & S_4 &= u_{2,3}+u_{3,2}, \\
S_2 &= u_{2,2}, & S_5 &= u_{1,3}+u_{3,1}, \\
S_3 &= u_{3,3}, & S_6 &= u_{1,2}+u_{2,1},
\end{aligned} \tag{1}
$$

the stress–strain relations for monoclinic symmetry, with the digonal axis x_1 and the axis normal to the plate x_2, are, for $c_{ij} = c_{ji}$,

$$
\begin{aligned}
T_1 &= c_{11}S_1+c_{12}S_2+c_{13}S_3+c_{14}S_4, \\
T_2 &= c_{21}S_1+c_{22}S_2+c_{23}S_3+c_{24}S_4, \\
T_3 &= c_{31}S_1+c_{32}S_2+c_{33}S_3+c_{34}S_4, \\
T_4 &= c_{41}S_1+c_{42}S_2+c_{43}S_3+c_{44}S_4, \\
T_5 &= c_{55}S_5+c_{56}S_6, \\
T_6 &= c_{65}S_5+c_{66}S_6.
\end{aligned} \tag{2}
$$

The stress equations of motion, with body forces omitted, are

$$
\begin{aligned}
T_{1,1}+T_{6,2}+T_{5,3} &= \rho\ddot{u}_1, \\
T_{6,1}+T_{2,2}+T_{4,3} &= \rho\ddot{u}_2, \\
T_{5,1}+T_{4,2}+T_{3,3} &= \rho\ddot{u}_3,
\end{aligned} \tag{3}
$$

where ρ is the mass per unit volume.

Substitution of (1) into (2) and then these results into (3) gives the displacement equations of motion for $u_i = u_i(x_1, x_2, t)$:

$$
\begin{aligned}
c_{11}u_{1,11}+c_{66}u_{1,22}+(c_{12}+c_{66})u_{2,12}+(c_{14}+c_{56})u_{3,12} &= \rho\ddot{u}_1, \\
(c_{12}+c_{66})u_{1,12}+c_{66}u_{2,11}+c_{22}u_{2,22}+c_{56}u_{3,11}+c_{24}u_{3,22} &= \rho\ddot{u}_2, \\
(c_{14}+c_{56})u_{1,12}+c_{56}u_{2,11}+c_{24}u_{2,22}+c_{55}u_{3,11}+c_{44}u_{3,22} &= \rho\ddot{u}_3.
\end{aligned} \tag{4}
$$

The electrode plating is assumed to contribute only inertia effects to the boundary conditions on the surfaces $x_2 = \pm h$. With the density of the plating ρ' and its thickness $2h' \ll h$, these boundary conditions, again for $u_i = u_i(x_1, x_2, t)$, are

$$
\begin{aligned}
T_6 &= c_{56}u_{3,1}+c_{66}(u_{1,2}+u_{2,1}) &&= \mp 2\rho' h' \ddot{u}_1, \\
T_2 &= c_{12}u_{1,1}+c_{22}u_{2,2}+c_{24}u_{3,2} &&= \mp 2\rho' h' \ddot{u}_2, \\
T_4 &= c_{14}u_{1,1}+c_{24}u_{2,2}+c_{44}u_{3,2} &&= \mp 2\rho' h' \ddot{u}_3,
\end{aligned} \tag{5}
$$

on $x_2 = \pm h$. Equations (4) and (5) are required to obtain the solutions of an infinite, plated plate. For the unplated case, the equations of (5) reduce to the traction-free conditions: $T_6 = T_2 = T_4 = 0$ on $x_2 = \pm h$

At the interfaces $x_1 = \pm a$, between plated and unplated portions, continuity of displacement and traction across the common surface must be satisfied:

$$\begin{aligned} u_1 &= \bar{u}_1, & u_2 &= \bar{u}_2, & u_3 &= \bar{u}_3, \\ T_1 &= \bar{T}_1, & T_5 &= \bar{T}_5, & T_6 &= \bar{T}_6, \end{aligned} \tag{6}$$

where barred symbols pertain to the plated portion. By applying (6) and both infinite plate solutions, we can obtain the solution of the partially plated plate shown in Fig. 1.

CHARACTERISTIC EQUATION FOR AN INFINITE, PLATED PLATE

In order to furnish the information required for the determination of Bechmann's Number, an asymptotic solution of (4) and (5), in terms of frequency as a function of the wave number (ξ) in the x_1 direction, is obtained for the infinite, plated plate. We take, as appropriate forms,

$$\begin{aligned} \bar{u}_1 &= \bar{A}_1 \cos\xi x_1 \sin\bar{\eta} x_2\, \mathrm{e}^{i\bar{\omega}t}, \\ \bar{u}_2 &= \bar{A}_2 \sin\xi x_1 \cos\bar{\eta} x_2\, \mathrm{e}^{i\bar{\omega}t}, \\ \bar{u}_3 &= \bar{A}_3 \sin\xi x_1 \cos\bar{\eta} x_2\, \mathrm{e}^{i\bar{\omega}t}. \end{aligned} \tag{7}$$

Substitution of these expressions for $\bar{u}_i$ into (4) yields

$$\begin{aligned} (c_{11}\xi^2 + c_{66}\bar{\eta}^2 - \rho\bar{\omega}^2)\bar{A}_1 + (c_{12}+c_{66})\xi\bar{\eta}\bar{A}_2 + (c_{14}+c_{56})\xi\bar{\eta}\bar{A}_3 &= 0, \\ (c_{12}+c_{66})\xi\bar{\eta}\bar{A}_1 + (c_{66}\xi^2 + c_{22}\bar{\eta}^2 - \rho\bar{\omega}^2)\bar{A}_2 + (c_{56}\xi^2 + c_{24}\bar{\eta}^2)\bar{A}_3 &= 0, \\ (c_{14}+c_{56})\xi\bar{\eta}\bar{A}_1 + (c_{56}\xi^2 + c_{24}\bar{\eta}^2)\bar{A}_2 + (c_{55}\xi^2 + c_{44}\bar{\eta}^2 - \rho\bar{\omega}^2)\bar{A}_3 &= 0. \end{aligned} \tag{8}$$

Hence, for the assumed $\bar{u}_i$ to be solutions of (4), $\bar{\omega}$, ξ, and $\bar{\eta}$ must satisfy the characteristic equation

$$\begin{vmatrix} c_{11}\xi^2 + c_{66}\bar{\eta}^2 - \rho\bar{\omega}^2 & (c_{12}+c_{66})\xi\bar{\eta} & (c_{14}+c_{56})\xi\bar{\eta} \\ (c_{12}+c_{66})\xi\bar{\eta} & c_{66}\xi^2 + c_{22}\bar{\eta}^2 - \rho\bar{\omega}^2 & (c_{56}\xi^2 + c_{24}\bar{\eta}^2) \\ (c_{14}+c_{56})\xi\bar{\eta} & (c_{56}\xi^2 + c_{24}\bar{\eta}^2) & c_{55}\xi^2 + c_{44}\bar{\eta}^2 - \rho\bar{\omega}^2 \end{vmatrix} = 0. \tag{9}$$

This equation is identical in form with that obtained for the unplated case [2, 10]. This is to be expected because nothing in the development thus far depends on the nature of the boundary conditions. Examination of (9) reveals that there are three wave numbers $\bar{\eta}_i$, $i = 1, 2, 3$, for given values of $\bar{\omega}$ and ξ. Then, for $i = 1, 2, 3$, we have, from (8),

$$\begin{aligned} (c_{11}\bar{\beta}_i^2 + c_{66} - \rho\bar{\omega}^2/\bar{\eta}_i^2)\bar{A}_{1i} + (c_{12}+c_{66})\bar{\beta}_i\bar{A}_{2i} + (c_{14}+c_{56})\bar{\beta}_i\bar{A}_{3i} &= 0, \\ (c_{12}+c_{66})\bar{\beta}_i\bar{A}_{1i} + (c_{66}\bar{\beta}_i^2 + c_{22} - \rho\bar{\omega}^2/\bar{\eta}_i^2)\bar{A}_{2i} + (c_{56}\bar{\beta}_i^2 + c_{24})\bar{A}_{3i} &= 0, \\ (c_{14}+c_{56})\bar{\beta}_i\bar{A}_{1i} + (c_{56}\bar{\beta}_i^2 + c_{24})\bar{A}_{2i} + (c_{56}\bar{\beta}_i^2 + c_{44} - \rho\bar{\omega}^2/\bar{\eta}_i^2)\bar{A}_{3i} &= 0, \end{aligned} \tag{10}$$

where $\bar{\beta}_i = \xi/\bar{\eta}_i$. Also, the solutions must take the form

$$\begin{aligned} \bar{u}_1 &= \cos\xi x_1(\bar{A}_{11}\sin\bar{\eta}_1 x_2 + \bar{A}_{12}\sin\bar{\eta}_2 x_2 + \bar{A}_{13}\sin\bar{\eta}_3 x_2)\,\mathrm{e}^{i\bar{\omega}t}, \\ \bar{u}_2 &= \sin\xi x_1(\bar{A}_{21}\cos\bar{\eta}_1 x_2 + \bar{A}_{22}\cos\bar{\eta}_2 x_2 + \bar{A}_{23}\cos\bar{\eta}_3 x_2)\,\mathrm{e}^{i\bar{\omega}t}, \\ \bar{u}_3 &= \sin\xi x_1(\bar{A}_{31}\cos\bar{\eta}_1 x_2 + \bar{A}_{32}\cos\bar{\eta}_2 x_2 + \bar{A}_{33}\cos\bar{\eta}_3 x_2)\,\mathrm{e}^{i\bar{\omega}t}, \end{aligned} \tag{11}$$

or, more concisely,

$$\bar{u}_1 = e^{i\bar{\omega}t}\cos\bar{\xi}x_1 \sum_{j=1}^{3} \bar{L}_{1j}\bar{A}_{jj}\sin\bar{\eta}_j x_2,$$

$$\bar{u}_2 = e^{i\bar{\omega}t}\sin\bar{\xi}x_1 \sum_{j=1}^{3} \bar{L}_{2j}\bar{A}_{jj}\cos\bar{\eta}_j x_2, \tag{12}$$

$$\bar{u}_3 = e^{i\bar{\omega}t}\sin\bar{\xi}x_1 \sum_{j=1}^{3} \bar{L}_{3j}\bar{A}_{jj}\cos\bar{\eta}_j x_2,$$

where the amplitude ratios

$$\bar{L}_{ij} = \bar{A}_{ij}/\bar{A}_{jj}, \quad \text{(no summation is implied)} \tag{13}$$

are so chosen as to avoid indeterminacies at $\bar{\xi} = 0$ [10].

For $\bar{\xi} = 0$, (9) becomes

$$\begin{vmatrix} c_{66}\bar{\eta}^2 - \rho\bar{\omega}^2 & 0 & 0 \\ 0 & c_{22}\bar{\eta}^2 - \rho\bar{\omega}^2 & c_{24}\bar{\eta}^2 \\ 0 & c_{24}\bar{\eta}^2 & c_{44}\bar{\eta}^2 - \rho\bar{\omega}^2 \end{vmatrix} = 0. \tag{14}$$

Solutions of (14) are

$$\rho\bar{\omega}^2 = c_i\bar{\eta}^2, \qquad i = 1, 2, 3, \tag{15}$$

where

$$c_1 = c_{66},$$

$$c_{2,3} = \tfrac{1}{2}\{c_{22} + c_{44} \pm [(c_{22} - c_{44})^2 - 4c_{24}^2]^{\frac{1}{2}}\}.$$

The solutions of (9), valid for small $\bar{\beta}_i$, give the characteristic equation

$$\frac{\rho\bar{\omega}^2}{c_i\bar{\eta}_i^2} = 1 + \frac{k_i}{v_i^2}\bar{\beta}_i^2 + O(\bar{\beta}_i^4), \qquad i = 1, 2, 3, \tag{16}$$

where $v_i^2 = c_i/c_1$ and

$$k_1 = \frac{c_{11}}{c_1} - \frac{[(c_{12}+c_{66})\cos\theta + (c_{14}+c_{56})\sin\theta]^2}{c_1(c_2 - c_1)} - \frac{[(c_{14}+c_{56})\cos\theta - (c_{12}+c_{66})\sin\theta]^2}{c_1(c_3 - c_1)},$$

$$k_2 = \frac{c_{66}\cos^2\theta + 2c_{56}\sin\theta\cos\theta + c_{55}\sin^2\theta}{c_1} + \frac{[(c_{12}+c_{66})\cos\theta + (c_{14}+c_{56})\sin\theta]^2}{c_1(c_2 - c_1)},$$

$$k_3 = \frac{c_{55}\cos^2\theta - 2c_{56}\sin\theta\cos\theta + c_{66}\sin^2\theta}{c_1} + \frac{[(c_{14}+c_{56})\cos\theta - (c_{12}+c_{66})\sin\theta]^2}{c_1(c_3 - c_1)}.$$

The angle $\theta = \tan^{-1}[c_{24}/(c_2 - c_{44})]$, measured from the x_2 axis in the $x_2 - x_3$ plane, gives the inclination of the normal coordinates x_2' and x_3' when $\bar{\xi} = 0$. The x_1 axis completes the system of normal coordinates at $\bar{\xi} = 0$. Equation (16), valid in either the plated or unplated case, is another form of (24) in Reference [10].

By means of (16), the frequency can now be eliminated from the amplitude coefficients given by (10) and (13):

$$\begin{aligned}
\bar{L}_{21} &= L'_{21}\bar{\beta}_1 + O(\bar{\beta}_1^3),\\
\bar{L}_{31} &= L'_{31}\bar{\beta}_1 + O(\bar{\beta}_1^3),\\
\bar{L}_{12} &= L'_{12}\bar{\beta}_2 + O(\bar{\beta}_2^3),\\
\bar{L}_{13} &= L'_{13}\bar{\beta}_3 + O(\bar{\beta}_3^3),\\
\bar{L}_{23} &= -\tan\theta + O(\bar{\beta}_3^2),\\
\bar{L}_{32} &= \tan\theta + O(\bar{\beta}_2^2).
\end{aligned} \tag{17}$$

The L'_{ij}, identical with those of (26) in Ref. [10], are:

$$L'_{21} = \frac{(c_{12}+c_1)(c_1-c_{44})+(c_{14}+c_{56})c_{24}}{(c_2-c_1)(c_3-c_1)},$$

$$L'_{31} = \frac{(c_{12}+c_1)c_{24}+(c_{14}+c_{56})(c_1-c_{22})}{(c_2-c_1)(c_3-c_1)},$$

$$L'_{12} = \frac{(c_{12}+c_1)(c_2-c_{44})+c_{24}(c_{14}+c_{56})}{(c_2-c_1)(c_2-c_{44})},$$

$$L'_{13} = \frac{(c_{14}+c_{56})(c_3-c_{22})+(c_{12}+c_1)c_{24}}{(c_3-c_1)(c_3-c_{22})}.$$

SIMPLE THICKNESS-SHEAR MODES OF AN INFINITE, PLATED PLATE

Simple thickness modes, first studied by Koga [11] for an unplated plate, determine the possible frequencies at infinite wave length ($\bar{\xi} = 0$) in an infinite plate. These frequencies are the ordinates at which the branches of the dispersion relation intersect the frequency axis of the dispersion diagram. Needed in this paper is a result similar to Koga's, but for a plated plate. Three types of antisymmetric modes are possible, each identified with one of the three material constants c_i. For the considerations of this paper, only the thickness-shear modes are of interest, i.e. solutions of the type

$$\begin{aligned}
\bar{u}_1 &= \bar{A}\sin\bar{\eta}x_2\, e^{i\bar{\omega}t},\\
\bar{u}_2 &\equiv \bar{u}_3 \equiv 0.
\end{aligned} \tag{18}$$

This problem is identical with that previously solved for simple thickness-twist modes in a plated plate [12]. Substitution of (18) into (4) gives

$$\rho\bar{\omega}^2 = c_1\bar{\eta}^2, \qquad (c_1 = c_{66}) \tag{19}$$

which is identical with (15) for the case $i = 1$. Consideration of the boundary conditions (5) shows that, for this case, $\bar{T}_2 \equiv \bar{T}_4 \equiv 0$ and

$$c_1\bar{\eta}\cos\bar{\eta}h = 2\rho' h'\bar{\omega}^2\sin\bar{\eta}h. \tag{20}$$

Combination of (19) and (20) gives

$$R\bar{\eta}h \tan \bar{\eta}h = 1, \tag{21}$$

where $R = 2\rho' h'/\rho h$: the ratio of the plate-back mass to the mass of the quartz. The solution of (21) can be expressed as

$$\bar{\eta}_1^0 h = (n-\alpha_n)\pi/2, \qquad n = 1, 3, 5, \ldots \tag{22}$$

where α_n, dependent on R and n, is small in comparison with unity, for small Rn, and is zero for $R = 0$, corresponding to the unplated case. Equation (22) indicates that, as in the unplated case, the $\bar{u}_1$ displacement, antisymmetric through the thickness, has n nodes. Equations (19) and (22) give, for the cut-off frequencies,

$$\bar{\omega}_n^0 = \frac{(n-\alpha_n)\pi}{2h}\sqrt{\frac{c_1}{\rho}}. \tag{23}$$

In (22) and (23), the superscript "0" indicates that these are values at $\xi = 0$. The superscript "1" in $\bar{\eta}_1^0 h$ denotes a thickness-shear mode.

If the frequencies are non-dimensionalized by dividing them by the frequency of the lowest simple thickness-shear mode in an unplated plate $(\omega_1^0 = \pi(c_1/\rho)^{\frac{1}{2}}/2h)$, then

$$\begin{aligned} \Omega_n^0 &= \omega_n^0/\omega_1^0 = n, \\ \bar{\Omega}_n^0 &= \bar{\omega}_n^0/\omega_1^0 = n-\alpha_n. \end{aligned} \tag{24}$$

Solution of (21) and (22) for α_n gives

$$\alpha_n = Rn \tag{25}$$

for small values of Rn.

FREQUENCY EQUATION OF AN INFINITE, PLATED PLATE

Having obtained the frequencies corresponding to the simple thickness-shear modes from (23), we find the frequency equation of an infinite plate in the neighborhood of $\xi = 0$ by considering

$$\bar{\omega} = \frac{(n-\alpha_n+\bar{\varepsilon})\pi}{2h}\sqrt{\frac{c_1}{\rho}} \tag{26}$$

where $\bar{\varepsilon}$ is a function of ξ. Substitution of (26) in (16) gives the wave numbers through the plate thickness:

$$\bar{\eta}_i h = \frac{(n-\alpha_n)\pi}{2v_i} + \bar{\gamma}_i + O(\xi h \bar{\beta}_i^3), \qquad \begin{aligned} i &= 1, 2, 3, \\ n &= 1, 3, 5, \ldots \end{aligned} \tag{27}$$

where

$$\bar{\gamma}_i = \frac{\pi\bar{\varepsilon}}{2v_i} - \frac{k_i \xi^2 h^2}{(n-\alpha_n)\pi v_i}.$$

Equation (27) is valid for small $\bar{\beta}_i$ and it is assumed, tentatively, that $\bar{\varepsilon} = O(\xi h \bar{\beta}_i)$.

The boundary conditions for the plated plate have been given in (5). Substitution of (12) in these equations gives

$$\begin{aligned}
\sum_{j=1}^{3} \bar{A}_{jj}\bar{B}_{6j}\cos\bar{\eta}_j h &= 2\rho' h'\bar{\omega}^2 \sum_{j=1}^{3} \bar{A}_{jj}\bar{L}_{1j}\sin\bar{\eta}_j h,\\
-\sum_{j=1}^{3} \bar{A}_{jj}\bar{B}_{2j}\sin\bar{\eta}_j h &= 2\rho' h'\bar{\omega}^2 \sum_{j=1}^{3} \bar{A}_{jj}\bar{L}_{2j}\cos\bar{\eta}_j h,\\
-\sum_{j=1}^{3} \bar{A}_{jj}\bar{B}_{4j}\sin\bar{\eta}_j h &= 2\rho' h'\bar{\omega}^2 \sum_{j=1}^{3} \bar{A}_{jj}\bar{L}_{3j}\cos\bar{\eta}_j h,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\bar{B}_{6j} &= c_{66}\bar{\eta}_j\bar{L}_{1j} + c_{66}\xi\bar{L}_{2j} + c_{56}\xi\bar{L}_{3j},\\
\bar{B}_{2j} &= c_{12}\xi\bar{L}_{1j} + c_{22}\bar{\eta}_j\bar{L}_{2j} + c_{24}\bar{\eta}_j\bar{L}_{3j},\\
\bar{B}_{4j} &= c_{14}\xi\bar{L}_{1j} + c_{24}\bar{\eta}_j\bar{L}_{2j} + c_{44}\bar{\eta}_j\bar{L}_{3j}.
\end{aligned} \tag{29}$$

Using (17), we obtain values of $\bar{B}_{ij}$, valid for small $\bar{\beta}_i$:

$$\begin{aligned}
\bar{B}_{61} &= \bar{\eta}_1 c_1[1 + O(\bar{\beta}_1^2)],\\
\bar{B}_{62} &= \bar{\eta}_2 c_2[B'_{62}\bar{\beta}_2 + O(\bar{\beta}_2^3)],\\
\bar{B}_{63} &= \bar{\eta}_3 c_3[B'_{63}\bar{\beta}_3 + O(\bar{\beta}_3^3)],\\
\bar{B}_{21} &= \bar{\eta}_1 c_1[B'_{21}\bar{\beta}_1 + O(\bar{\beta}_1^3)],\\
\bar{B}_{22} &= \bar{\eta}_2 c_2[1 + O(\bar{\beta}_2^2)],\\
\bar{B}_{23} &= \bar{\eta}_3 c_3[-\tan\theta + O(\bar{\beta}_3^2)],\\
\bar{B}_{41} &= \bar{\eta}_1 c_1[B'_{41}\bar{\beta}_1 + O(\bar{\beta}_1^3)],\\
\bar{B}_{42} &= \bar{\eta}_2 c_2[\tan\theta + O(\bar{\beta}_2^2)],\\
\bar{B}_{43} &= \bar{\eta}_3 c_3[1 + O(\bar{\beta}_3^2)],
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
c_2 B'_{62} &= c_{56}\tan\theta + c_{66} + c_{66}L'_{12},\\
c_3 B'_{63} &= c_{56} - c_{66}\tan\theta + c_{66}L'_{13},\\
c_1 B'_{21} &= c_{12} + c_{22}L'_{21} + c_{24}L'_{31},\\
c_1 B'_{41} &= c_{14} + c_{24}L'_{21} + c_{44}L'_{31}.
\end{aligned}$$

From (28) we require

$$\bar{\Delta} \equiv \begin{vmatrix} \bar{B}_{6j}\cos\bar{\eta}_j h - \bar{M}_{1j}\sin\bar{\eta}_j h \\ \bar{B}_{2j}\sin\bar{\eta}_j h + \bar{M}_{2j}\cos\bar{\eta}_j h \\ \bar{B}_{4j}\sin\bar{\eta}_j h + \bar{M}_{3j}\cos\bar{\eta}_j h \end{vmatrix} = 0 \tag{31}$$

where the $\bar{M}_{ij}$ ($= 2\rho' h'\bar{\omega}^2\bar{L}_{ij}$) are the contributions of the plating inertia.

Solutions of (31), valid for small $\bar{\beta}_i$, can be obtained through the use of (17), (26), (27) and (30) as follows. Substitution of (27) in $\sin \bar{\eta}_i h$ and $\cos \bar{\eta}_i h$ gives

$$
\begin{aligned}
\sin \bar{\eta}_i h &= (1-\tfrac{1}{2}\bar{\gamma}_i^2)\sin \bar{G}_i + \bar{\gamma}_i \cos \bar{G}_i + O(\xi^3 h^3 \bar{\beta}_i^3) \\
\cos \bar{\eta}_i h &= (1-\tfrac{1}{2}\bar{\gamma}_i^2)\cos \bar{G}_i - \bar{\gamma}_i \sin \bar{G}_i + O(\xi^3 h^3 \bar{\beta}_i^3)
\end{aligned} \tag{32}
$$

where $\bar{G}_i = (n-\alpha_n)\pi/2v_i$. The error involved in $\sin \bar{\eta}_i h$ is changed to $O(\xi^4 h^4 \bar{\beta}_i^4)$ if $\cos \bar{G}_i \approx 0$ and, when $\sin \bar{G}_i \approx 0$ the error becomes $O(\xi^4 h^4 \bar{\beta}^4)$ in $\cos \bar{\eta}_i h$. The latter possibility is of importance in some of the higher modes ($n > 5$) for the AT-cut of quartz. This condition occurs when another antisymmetric branch, particularly thickness-stretch, is close to the thickness-shear branch in the neighborhood of the frequency axis.

Substitution of (17) and (26) in $\bar{M}_{ij}$, gives

$$
\begin{aligned}
\bar{M}_{1j} &= \bar{\eta}_j c_j R \bar{G}_j [L'_{1j} + O(\bar{\beta}_j^2)], \qquad j = 2, 3, \\
\bar{M}_{j1} &= \bar{\eta}_1 c_1 R \bar{G}_1 [L'_{j1} + O(\bar{\beta}_j^2)], \qquad j = 2, 3, \\
\bar{M}_{32} &= \bar{\eta}_2 c_2 R \bar{G}_2 [\tan\theta + O(\bar{\beta}_2^2)], \\
\bar{M}_{23} &= \bar{\eta}_3 c_3 R \bar{G}_3 [-\tan\theta + O(\bar{\beta}_3^2)], \\
\bar{M}_{jj} &= \bar{\eta}_j c_j R \bar{G}_j [1 + O(\bar{\beta}_j^2)], \qquad j = 1, 2, 3.
\end{aligned} \tag{33}
$$

Finally, by use of (30), (32) and (33), the frequency equation, valid for $Rn \ll 1, \bar{\varepsilon} = O(\xi h \bar{\beta}_i)$ and small $\bar{\beta}_i$, is obtained from (31):

$$
\begin{aligned}
\bar{\Delta} = & -\bar{\gamma}_1[\sin \bar{G}_2 + (R\bar{G}_2 + \bar{\gamma}_2)\cos \bar{G}_2][\sin \bar{G}_3 + (R\bar{G}_3 + \bar{\gamma}_3)\cos \bar{G}_3](\cos\theta)^{-2} \\
& \times [1 + O(R) + O(R^2 n^2)] + v_3 \psi^2 [\sin \bar{G}_2 + R\bar{G}_2 \cos \bar{G}_2](B'_{63}\cos \bar{G}_3 - R\bar{G}_3 L'_{13} \sin \bar{G}_3) \\
& \times (B'_{21}\tan\theta - B'_{41})[1 + O(R^2 n^2) + O(\bar{\gamma}_2)] - v_2 \psi^2 [\sin \bar{G}_3 + R\bar{G}_3 \cos \bar{G}_3] \\
& \times (B'_{62}\cos \bar{G}_2 - R\bar{G}_2 L'_{12} \sin \bar{G}_2)(B'_{41}\tan\theta + B'_{21})[1 + O(R^2 n^2) + O(\bar{\gamma}_3)] \\
= & \; 0
\end{aligned} \tag{34}
$$

where $\psi = 2\xi h/(n-\alpha_n)\pi$.

Equation (34), for a particular mode number n and plate-back ratio R, is a quadratic equation in $\bar{\varepsilon}$ and $\xi^2 h^2$ if the product $\bar{\gamma}_1 \bar{\gamma}_2 \bar{\gamma}_3$ is dropped. It represents a thickness-shear branch in the regions of small $\bar{\beta}_i$ and shows that the branch has zero slope at $\bar{\varepsilon} = \xi = 0$. This frequency equation is valid for any monoclinic plate with thin, uniform plating of negligible stiffness for which $Rn \ll 1$.

If coupling of this thickness-shear mode with another, e.g. thickness-stretch, is not too great,

$$
\begin{aligned}
\sin \bar{G}_2 + (R\bar{G}_2 + \bar{\gamma}_2)\cos \bar{G}_2 &\approx \sin G_2, \\
\sin \bar{G}_3 + (R\bar{G}_3 + \bar{\gamma}_3)\cos \bar{G}_3 &\approx \sin G_3,
\end{aligned} \tag{35}
$$

and then (34) reduces to an equation in $\bar{\varepsilon}$ and $\xi^2 h^2$ only, i.e. a parabola. In this form the branch, at small $\bar{\beta}_i$, is fully described by its ordinate, slope (equal to zero) and curvature at $\xi = 0$. Whether this simplification can be adopted is dependent upon the relative magnitudes of the elastic constants and the particular mode number n in question. For the AT-cut the simplified frequency equation gives a sufficiently precise description of the branches at long wave lengths (accurate to one unit in the third significant figure) for all but the ninth ($n = 17$) of the first ten modes if the following elastic constants, computed from the principal values for alpha-quartz established by Bechmann [13], are used:

$$\begin{array}{lll} c_{11} = 86{\cdot}74 & c_{55} = 68{\cdot}81 & c_{66} = 29{\cdot}01 \\ c_{12} = -8{\cdot}25 & c_{56} = 2{\cdot}53 & \\ c_{14} = -3{\cdot}66 & c_{22} = 129{\cdot}77 & \\ c_{24} = 5{\cdot}70 & c_{44} = 38{\cdot}61 & (\times 10^{10}\ \text{dyn-cm}^{-2}). \end{array}$$

For $n = 17$, the thickness-shear branch is greatly influenced by the proximity of the fourth antisymmetric thickness-stretch branch. The assumption of the first of (35) is, in that case, not valid.

If $\sin \bar{G}_2 + R\bar{G}_2 \cos \bar{G}_2 = 0$, the thickness-shear branch coincides with an antisymmetric thickness-stretch branch and the slope of the branch at $\bar{\xi} = 0$ is no longer zero. This is the situation which is approached at $n = 17$ for the AT-cut.

Likewise, if $\sin \bar{G}_3 + R\bar{G}_3 \cos \bar{G}_3 = 0$, coincidence with a face-shear branch occurs. In the AT-cut, the thickness-shear branch can approach a branch of this type much more closely than a thickness-stretch branch before the shapes of the branches are greatly affected. This is due to the very weak coupling to face-shear through the elastic constants for the AT-cut.

When $\bar{\varepsilon}$ is in the neighborhood of α_n, i.e. when the frequency in the infinite, plated plate is near the cut-off frequency of the unplated plate, (34) reduces with the aid of (25), to

$$\begin{aligned} \bar{\Delta} = & -\bar{\gamma}_1\left[\sin\frac{n\pi}{2v_2}\sin\frac{n\pi}{2v_3}+\bar{\gamma}_2\cos\frac{n\pi}{2v_2}\sin\frac{n\pi}{2v_3}+\bar{\gamma}_3\cos\frac{n\pi}{2v_3}\sin\frac{n\pi}{2v_2}\right][1+O(R)+O(R^2n^2)] \\ & +\left[v_3\left(\frac{2E_3\bar{\xi}h}{n\pi}\right)^2\cos\frac{n\pi}{2v_3}\sin\frac{n\pi}{2v_2}+v_2\left(\frac{2E_2\bar{\xi}h}{n\pi}\right)^2\cos\frac{n\pi}{2v_2}\sin\frac{n\pi}{2v_3}\right][1+O(Rn)] = 0 \end{aligned} \tag{36}$$

where

$$E_2 = \frac{(c_2+c_{12})\cos\theta+(v_2^2c_{56}+c_{14})\sin\theta}{v_2(c_2-c_1)},$$

$$E_3 = \frac{(v_3^2c_{56}+c_{14})\cos\theta-(c_3+c_{12})\sin\theta}{v_3(c_3-c_1)}.$$

Equation (36) is identical in form with the frequency equation for the unplated branch. In the notation of Ref. [10], $E_2^2 = \mathbf{E}/\mathbf{B}$ and $E_3^2 = \mathbf{F}/\mathbf{B}$. Thus for all practical purposes, the effect of this amount of plating ($Rn \ll 1$) is to shift the branches down the frequency axis an amount proportional to α_n (Fig. 2), but the shape remains the same. When the approximations of (35) are valid and $\bar{\varepsilon}$ is in the neighborhood of α_n, (34) reduces further to

$$\bar{\varepsilon} = K_n^2\bar{\xi}^2h^2 + O(R^2n), \tag{37}$$

where

$$K_n = \left[\frac{2k_1}{n\pi}+\frac{8E_2^2v_2}{n^2\pi^3}\cot\frac{n\pi}{2v_2}+\frac{8E_3^2v_3}{n^2\pi^3}\cot\frac{n\pi}{2v_3}\right]^{\frac{1}{2}}. \tag{38}$$

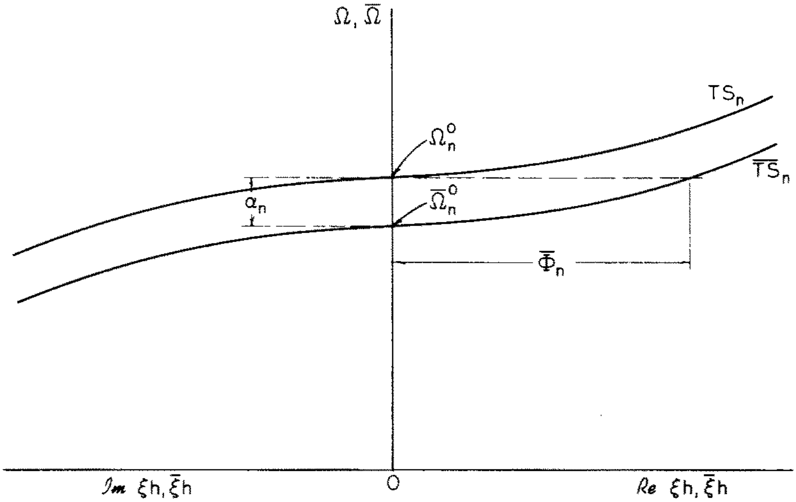

FIG. 2. Typical detail of the branches of the dispersion relation for thickness-shear modes of order n in an infinite plated plate ($\overline{TS}_n$) and unplated plate (TS_n).

In the subsequent development, all branches are ignored except that thickness-shear branch in the vicinity of the frequency axis for a particular mode number n. For this simplification to be valid, two conditions must be satisfied. First, K_n^2 must not be negative. If it were negative, another branch would be above and sufficiently close to the thickness-shear branch to cause a decrease in frequency for increasing real $\bar{\xi}h$. Bechmann's Number does not exist in such a situation. The second condition is that

$$\alpha_n < |n-2mv_j|, \qquad j = 2, 3, m \text{ is any integer.} \tag{39}$$

Satisfaction of (39) insures that neither a thickness-stretch nor a face-shear branch is present between $\bar{\Omega}_n^0$ and Ω_n^0 at wave numbers near the frequency axis.

With these conditions the displacement solutions, omitting the time factor $e^{i\bar{\omega}t}$, can be written in the form

$$\begin{aligned} \bar{u}_1 &= \bar{A}_{11} \cos \bar{\xi} x_1 \sum_{j=1}^{3} \bar{L}_{1j}\bar{\rho}_j \sin \bar{\eta}_j x_2, \\ \bar{u}_2 &= \bar{A}_{11} \sin \bar{\xi} x_1 \sum_{j=1}^{3} \bar{L}_{2j}\bar{\rho}_j \cos \bar{\eta}_j x_2, \\ \bar{u}_3 &= \bar{A}_{11} \sin \bar{\xi} x_1 \sum_{j=1}^{3} \bar{L}_{3j}\bar{\rho}_j \cos \bar{\eta}_j x_2, \end{aligned} \tag{40}$$

where $\bar{\xi}$, which obeys (34), or (37) when appropriate, is the abscissa of the one thickness-shear branch in the vicinity of the frequency axis. The infinity of other branches existing at frequency $\bar{\omega}$ are ignored. The amplitude ratios

$$\bar{\rho}_j = \bar{A}_{jj}/\bar{A}_{11}, \tag{41}$$

are found from (28) to be

$$\bar{\rho}_j = \bar{\rho}_j'\bar{\beta}_j + O(R^2n^2\bar{\beta}_j), \qquad j = 2, 3, \tag{42}$$

where, under the restriction of (39),

$$
\begin{aligned}
\bar{\rho}'_2 &= -\frac{(-1)^{[(n-1)/2]}\cos\theta E_2}{\sin(n\pi/2v_2)},\\
\bar{\rho}'_3 &= \frac{(-1)^{[(n-1)/2]}\cos\theta E_3}{\sin(n\pi/2v_3)}.
\end{aligned} \tag{43}
$$

These $\bar{\rho}'_j$ are identical with the corresponding ρ'_j of the unplated case.

ANHARMONIC OVERTONES OF A PARTIALLY PLATED PLATE

At frequencies below cut-off in an infinite, unplated plate, solutions of the type presented for the plated case are obtained in which the wave number ξ is imaginary, indicating a non-propagating wave. For the partially plated plate illustrated in Fig. 1, the two infinite plate solutions are employed, and the conditions at the interface $x_1 = \pm a$, given in equations (6), are imposed. The frequency range of interest at some thickness-shear mode, represented by n, is the narrow band between the two frequencies

$$
\omega_n^0 = \frac{n\pi}{2}\sqrt{\frac{c_1}{\rho}} \quad \text{and} \quad \bar{\omega}_n^0 = \frac{(n-\alpha_n)\pi}{2}\sqrt{\frac{c_1}{\rho}}.
$$

Rather than use the formulation involving an imaginary wave number, we write the solutions for the unplated region in the form

$$
\begin{aligned}
u_1\begin{Bmatrix} x_1 > a \\ x_1 < -a \end{Bmatrix} &= A_{11}\exp[\mp\xi(x_1 \mp a)] \sum_{j=1}^{3} L_{1j}\rho_j \sin\eta_j x_2,\\
u_2\begin{Bmatrix} x_1 > a \\ x_1 < -a \end{Bmatrix} &= \pm A_{11}\exp[\mp\xi(x_1 \mp a)] \sum_{j=1}^{3} L_{2j}\rho_j \cos\eta_j x_2,\\
u_3\begin{Bmatrix} x_1 > a \\ x_1 < -a \end{Bmatrix} &= \pm A_{11}\exp[\mp\xi(x_1 \mp a)] \sum_{j=1}^{3} L_{3j}\rho_j \cos\eta_j x_2,
\end{aligned} \tag{44}
$$

in which ξ is real. The quantities L_{ij}, ρ_j, η_j are identical with their corresponding symbols for the plated case defined previously but with $\bar{\beta}_j \to \beta_j$, $\bar{\varepsilon} \to \varepsilon$, $\bar{\xi}^2 \to -\xi^2$, and $\alpha_n = R = 0$. Then (17), (27), and (42) give

$$
\begin{aligned}
u_1 &= A_{11}\exp[\mp\xi(x_1 \mp a)][1+O(\beta_1^2)]\sin\eta_1 x_2,\\
u_2 &= \pm A_{11}\exp[\mp\xi(x_1 \mp a)][1+O(\beta_1^2)]\left(\frac{2\xi}{n\pi}\right)u'_2,\\
u_3 &= \pm A_{11}\exp[\mp\xi(x_1 \mp a)][1+O(\beta_1^2)]\left(\frac{2\xi}{n\pi}\right)u'_3,
\end{aligned} \tag{45}
$$

where

$$
\begin{aligned}
u'_2 &= L'_{21}\cos\eta_1 x_2 + \rho'_2\cos\eta_2 x_2 - \tan\theta\,\rho'_3\cos\eta_3 x_2,\\
u'_3 &= L'_{31}\cos\eta_1 x_2 + \tan\theta\rho'_2\cos\eta_2 x_2 + \rho'_3\cos\eta_3 x_2.
\end{aligned}
$$

From (39) the plated quantities can be written as

$$\begin{aligned} \bar{u}_1 &= \bar{A}_{11}\cos\bar{\xi}x_1[1+\alpha_n EF(x_2)]\sin\eta_1 x_2, \\ \bar{u}_2 &= \bar{A}_{11}\sin\bar{\xi}x_1[1+\alpha_n EF(x_2)]\left(\frac{2\bar{\xi}}{n\pi}\right)u_2', \\ \bar{u}_3 &= \bar{A}_{11}\sin\bar{\xi}x_1[1+\alpha_n EF(x_2)]\left(\frac{2\bar{\xi}}{n\pi}\right)u_3', \end{aligned} \tag{46}$$

where $EF(x_2)$ designates an even function in x_2 with amplitude approximately equal to unity. Employing the displacement continuity condition of (6) and ignoring the error terms in (45) and (46), we obtain

$$\begin{aligned} A_{11} &= \bar{A}_{11}\cos\bar{\xi}a, \\ A_{11}\xi &= \bar{A}_{11}\bar{\xi}\sin\bar{\xi}a. \end{aligned} \tag{47}$$

Consideration of the traction continuity condition represented by the stress equalities of (6) also gives (47) to the same degree of approximation. Equations (47) require

$$\begin{vmatrix} 1 & \cos\bar{\xi}a \\ \xi & \bar{\xi}\sin\bar{\xi}a \end{vmatrix} = 0, \tag{48}$$

or

$$\frac{a}{h} = \frac{1}{\bar{\xi}h}\tan^{-1}\frac{\xi}{\bar{\xi}} \tag{49}$$

Note that (49) is identical with (41) of Reference [7] if the plating factors in that case are taken to be unity. Equation (49), the plated frequency equation of (34) and the unplated frequency equation taken from (36) ($\bar{\varepsilon} \to \varepsilon$, $\bar{\xi}^2 \to -\xi^2$) give the typical frequency spectrum illustrated in Fig. 3. In this analysis only one of the infinity of branches was considered when applying each infinite plate solution. Additional branches would have to be included to satisfy the continuity condition more precisely at the interface.

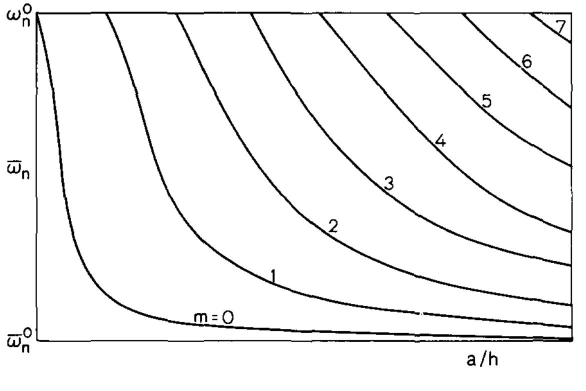

FIG. 3. Typical frequency spectrum of the thickness-shear anharmonic overtones of a partially plated plate between the cut-off frequencies for plated ($\bar{\omega}_n^0$) and unplated (ω_n^0) infinite plates.

BECHMANN'S NUMBER

It is apparent from Fig. 3 that the values of a/h for which one additional overtone is present may be found from (49) for $\xi = 0$ (the wave number at the cut-off frequency of an infinite, unplated plate). This gives the equation

$$\frac{a}{h} = \frac{m\pi}{\bar{\xi}h}\bigg|_{\xi=\alpha_n}, \qquad m = 0, 1, 2, 3, \ldots \tag{50}$$

The first overtone becomes possible when $m = 1$. For $\bar{\Phi}_n = \bar{\xi}h|_{\xi=\alpha_n}$ (Fig. 2), Bechmann's Number for the nth harmonic overtone of thickness-shear is therefore

$$B_n^{TS} = \pi/\bar{\Phi}_n. \tag{51}$$

Since $\bar{\xi} = 2\pi/L$, where L is the wave length in the plated portion, then $2a = L$. In other words, similar to the result found in Ref. [7] for the first harmonic thickness-shear mode, the critical plating width for any harmonic thickness-shear mode is equal to the wave length of that mode in the infinite, plated plate at the cut-off frequency of the corresponding mode in the unplated plate. This result was employed as a postulate in Ref. [8] for harmonic overtones of thickness-twist modes. It should also be noted that, in the case of the thickness-twist modes, the difficulty with near coincidence of cut-off frequencies of two types of modes does not occur.

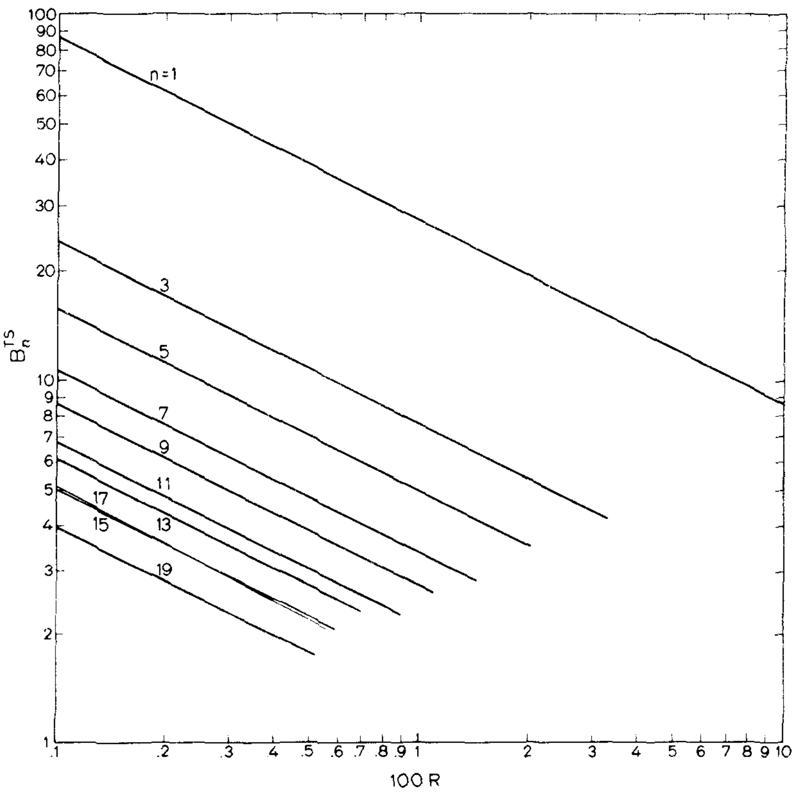

FIG. 4. Bechmann's Number (B_n^{TS}) vs. percent plate-back (100R) for thickness-shear modes of order n in the AT-cut quartz plate. ($Rn < 0{\cdot}1$).

Where the approximations of (35) are valid, (37) and (51) give

$$B_n^{TS} = \pi K_n(Rn)^{-\frac{1}{2}} \tag{52}$$

where K_n is given by (38).

When another antisymmetric branch is so close to the thickness-shear branch in the dispersion diagram that either one of (35) is invalid, then (36), rather than (37), with $\bar{\varepsilon} = \alpha_n = Rn$ must be used with (51) to get B_n^{TS}. In any case, K_n must be real and the condition of (39) must be satisfied. Equation (52) is valid for all except $n = 17$ of the first ten modes in the AT-cut.

Figure 4 gives the values of B_n^{TS} for the AT-cut with Bechmann's constants for quartz. It is necessary to restrict the values to $Rn \ll 1$ for (36) and (37) to be valid.

It should be remarked that the higher n becomes, the greater is the required accuracy of the elastic constants. This is particularly crucial when branches are close in the dispersion diagram of the infinite plate. In these instances, slight changes in the constants could radically affect the behavior of the branch. For example, a change of 0·007 in the value of v_2 could cause coincidence of the thickness-shear and thickness-stretch branches at $n = 17$ in the AT-cut.

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Абстракт—Рассматривается метод определения числа Бехмана для гармонических обертонов антисимметрических сдвиговых колебаний кристаллических пластинок. В большинстве случаев решение можно представить простой формулой в явном виде. Приводятся расчеты величины числа Бехмана кварцевой пластинки AT среза для первых десяти форм колебаний для пределов процентного отношения опирания пластинки, которые встречаются в применениях.